Integr. Equ. Oper. Theory
DOI 10.1007/s00020-017-2371-0

Integral Equations and Operator Theory

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Spectral Bounds for the Torsion Function

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Abstract. Let Ω be an open set in Euclidean space $\mathbb{R}^m$, $m = 2, 3, ...$, and let v_Ω denote the torsion function for Ω. It is known that v_Ω is bounded if and only if the bottom of the spectrum of the Dirichlet Laplacian acting in $\mathcal{L}^2(\Omega)$, denoted by $\lambda(\Omega)$, is bounded away from 0. It is shown that the previously obtained bound $\|v_\Omega\|_{\mathcal{L}^\infty(\Omega)}\lambda(\Omega) \ge 1$ is sharp: for $m \in \{2, 3, ...\}$, and any $\epsilon > 0$ we construct an open, bounded and connected set $\Omega_\epsilon \subset \mathbb{R}^m$ such that $\|v_{\Omega_\epsilon}\|_{\mathcal{L}^\infty(\Omega_\epsilon)}\lambda(\Omega_\epsilon) < 1+\epsilon$. An upper bound for v_Ω is obtained for planar, convex sets in Euclidean space $\mathbb{R}^2$, which is sharp in the limit of elongation. For a complete, non-compact, m-dimensional Riemannian manifold M with non-negative Ricci curvature, and without boundary it is shown that v_Ω is bounded if and only if the bottom of the spectrum of the Dirichlet–Laplace–Beltrami operator acting in $\mathcal{L}^2(\Omega)$ is bounded away from 0.

Mathematics Subject Classification. Primary 58J32, 58J35, 35K20.

Keywords. Torsion function, Dirichlet Laplacian, Riemannian manifold, Non-negative Ricci curvature.

1. Introduction

Let Ω be an open set in $\mathbb{R}^m$, and let Δ be the Laplace operator acting in $L^2(\mathbb{R}^m)$. Let $(B(s), s \ge 0; \mathbb{P}_x, x \in \mathbb{R}^m)$ be Brownian motion on $\mathbb{R}^m$ with generator Δ. For $x \in \Omega$ we denote the first exit time, and expected lifetime of Brownian motion by

$$T_\Omega = \inf\{s \ge 0 : B(s) \notin \Omega\},$$

and

$$v_\Omega(x) = \mathbb{E}_x[T_\Omega],\ x \in \Omega, \tag{1}$$

respectively, where $\mathbb{E}_x$ denotes the expectation associated with $\mathbb{P}_x$. Then v_Ω is the torsion function for Ω, i.e. the unique solution of

$$-\Delta v = 1,\ v \in H_0^1(\Omega). \tag{2}$$

The bottom of the spectrum of the Dirichlet Laplacian acting in $\mathcal{L}^2(\Omega)$ is denoted by

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Published online: 26 April 2017

$$\lambda(\Omega) = \inf_{\varphi \in H_0^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |D\varphi|^2}{\displaystyle\int_\Omega \varphi^2}. \tag{3}$$

It was shown in [1,2] that $\|v_\Omega\|_{\mathcal{L}^\infty(\Omega)}$ is finite if and only if $\lambda(\Omega) > 0$. Moreover, if $\lambda(\Omega) > 0$, then

$$\lambda(\Omega)^{-1} \le \|v_\Omega\|_{\mathcal{L}^\infty(\Omega)} \le (4 + 3m \log 2)\lambda(\Omega)^{-1}. \tag{4}$$

The upper bound in (4) was subsequently improved (see [3]) to

$$\|v_\Omega\|_{\mathcal{L}^\infty(\Omega)} \le \frac{1}{8}(m + cm^{1/2} + 8)\lambda(\Omega)^{-1},$$

where

$$c = (5(4 + \log 2))^{1/2}.$$

In Theorem 1 below we show that the coefficient 1 of $\lambda(\Omega)^{-1}$ in the left-hand side of (4) is sharp.

Theorem 1. *For $m \in \{2, 3, \dots\}$, and any $\epsilon > 0$ there exists an open, bounded, and connected set $\Omega_\epsilon \subset \mathbb{R}^m$ such that*

$$\|v_{\Omega_\epsilon}\|_{\mathcal{L}^\infty(\Omega_\epsilon)}\, \lambda(\Omega_\epsilon) < 1 + \epsilon. \tag{5}$$

The set Ω_ϵ is constructed explicitly in the proof of Theorem 1.

It has been shown by L. E. Payne (see (3.12) in [4]) that for any convex, open $\Omega \subset \mathbb{R}^m$ for which $\lambda(\Omega) > 0$,

$$\|v_\Omega\|_{\mathcal{L}^\infty(\Omega)}\, \lambda(\Omega) \ge \frac{\pi^2}{8}, \tag{6}$$

with equality if Ω is a slab, i.e. the connected, open set, bounded by two parallel $(m-1)$-dimensional hyperplanes. Theorem 2 below shows that for any sufficiently elongated, convex, planar set (not just an elongated rectangle) $\|v_\Omega\|_{\mathcal{L}^\infty(\Omega)}\lambda(\Omega)$ is approximately equal to $\frac{\pi^2}{8}$. We denote the width and the diameter of a bounded open set Ω by $w(\Omega)$ (i.e. the minimal distance of two parallel lines supporting Ω), and $\mathrm{diam}(\Omega) = \sup\{|x - y| : x \in \Omega, y \in \Omega\}$ respectively.

Theorem 2. *If Ω is a bounded, planar, open, convex set with width $w(\Omega)$, and diameter* $\mathrm{diam}(\Omega)$, *then*

$$\|v_\Omega\|_{\mathcal{L}^\infty(\Omega)}\, \lambda(\Omega) \le \frac{\pi^2}{8}\left(1 + 7 \cdot 3^{2/3}\left(\frac{w(\Omega)}{\mathrm{diam}(\Omega)}\right)^{2/3}\right).$$

In the Riemannian manifold setting we denote the bottom of the spectrum of the Dirichlet–Laplace–Beltrami operator by (3). We have the following.

Theorem 3. *Let M be a complete, non-compact, m-dimensional Riemannian manifold, without boundary, and with non-negative Ricci curvature. There* *exists $K < \infty$, depending on M only, such that if $\Omega \subset M$ is open, and $\lambda(\Omega) > 0$, then*

$$\lambda(\Omega)^{-1} \le \|v_\Omega\|_{\mathcal{L}^\infty(\Omega)} \le 2^{(3m+8)/4} \cdot 3^{m/2} K^2 \lambda(\Omega)^{-1}, \tag{7}$$

where K is the constant in the Li-Yau inequality in (35) below.

The proofs of Theorems 1, 2, and 3 will be given in Sects. 2, 3 and 4 respectively.

Below we recall some basic facts on the connection between torsion function and heat kernel. It is well known (see [5–7]) that the heat equation

$$\Delta u(x;t) = \frac{\partial u(x;t)}{\partial t}, \quad x \in M, \quad t > 0,$$

has a unique, minimal, positive fundamental solution $p_M(x,y;t)$, where $x \in M$, $y \in M$, $t > 0$. This solution, the heat kernel for M, is symmetric in x, y, strictly positive, jointly smooth in $x, y \in M$ and $t > 0$, and it satisfies the semigroup property

$$p_M(x,y;s+t) = \int_M dz\; p_M(x,z;s) p_M(z,y;t),$$

for all $x, y \in M$ and $t, s > 0$, where dz is the Riemannian measure on M. See, for example, [8] for details. If Ω is an open subset of M, then we denote the unique, minimal, positive fundamental solution of the heat equation on Ω by $p_\Omega(x,y;t)$, where $x \in \Omega, y \in \Omega, t > 0$. This Dirichlet heat kernel satisfies,

$$p_\Omega(x,y;t) \le p_M(x,y;t),\ x \in \Omega, y \in \Omega, t > 0.$$

Define $u_\Omega : \Omega \times (0,\infty) \mapsto \mathbb{R}$ by

$$u_\Omega(x;t) = \int_\Omega dy\, p_\Omega(x,y;t).$$

Then,

$$u_\Omega(x;t) = \mathbb{P}_x[T_\Omega > t],$$

and by (1)

$$v_\Omega(x) = \int_0^\infty dt\, \mathbb{P}_x\left[T_\Omega > t\right] = \int_0^\infty dt \int_\Omega dy\, p_\Omega(x,y;t). \tag{8}$$

It is straightforward to verify that v_Ω as in (8) satisfies (2).

2. Proof of Theorem 1

We introduce the following notation. Let $C_L = (-\frac{L}{2}, \frac{L}{2})^{m/2}$ be the open cube with measure L^m, and delete from C_L, N^m closed balls with radii δ, where each ball $B(c_i;\delta)$ is positioned at the centre of an open cube Q_i with measure $(L/N)^m$. These open cubes are pairwise disjoint, and contained in C_L. Let $0 < \delta < \frac{L}{2N}$, and put (Fig. 1)

$$\Omega_{\delta,N,L} = C_L - \cup_{i=1}^{N^m} B(c_i;\delta).$$

The set $\Omega_{\delta,\mathrm{N},\mathrm{L}}$ also features in [9], where the sharpness of an inequality due to Pólya has been established.

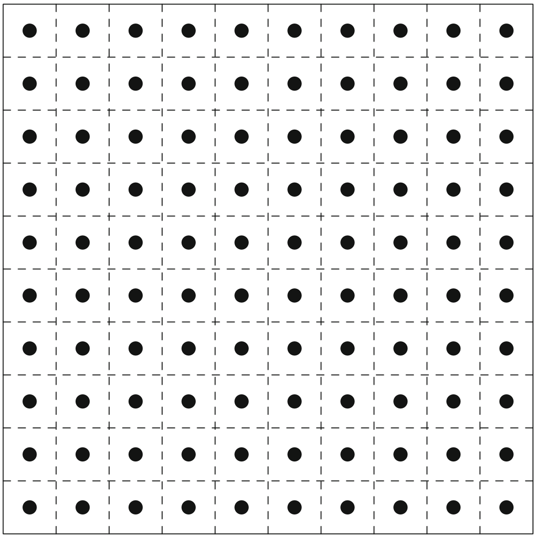

FIGURE 1. $\Omega_{\delta,N,L}$ with $m=2, N=10, \delta=\frac{L}{8N}$

Below we will show that for any $\epsilon>0$ we can choose δ, N such that

$$\left\|v_{\Omega_{\delta,N,L}}\right\|_{\mathcal{L}^\infty(\Omega_{\delta,N,L})}\lambda\left(\Omega_{\delta,N,L}\right)<1+\epsilon.$$

In Lemma 4 below we show that $\lambda(\Omega_{\delta,N,L})$ is approximately equal to the first eigenvalue, $\mu_{1,B(0;\delta),L/N}$, of the Laplacian with Neumann boundary conditions on $\partial C_{L/N}$, and with Dirichlet boundary conditions on $\partial B(0;\delta)$. The requirement $\mu_{1,B(0;\delta),L/N}$ not being too small stems from the fact that the approximation of replacing the Neumann boundary conditions on C_L is a surface effect which should not dominate the leading term $\mu_{1,B(0;\delta),L/N}$.

Lemma 4. *If* $\delta\le\frac{L}{4N}$, $N\ge 10$, *and* $\frac{N}{L^2}\le\mu_{1,B(0;\delta),L/N}$, *then*

$$\lambda\left(\Omega_{\delta,N,L}\right)\le\mu_{1,B(0;\delta),L/N}+32m\left(\frac{5}{4}\right)^m\left(\frac{N}{L^2}+\frac{1}{N^{1/2}}\mu_{1,B(0;\delta),L/N}\right).$$

Proof. Let $\varphi_{1,B(0;\delta),L/N}$ be the first eigenfunction (positive) corresponding to $\mu_{1,B(0;\delta),L/N}$, and normalised in $\mathcal{L}^2(C_{L/N}-B(0;\delta))$. In order to prove the lemma we construct a test function by periodically extending $\varphi_{1,B(0;\delta),L/N}$ to all cubes $Q_1,\dots Q_{N^m}$ of $\Omega_{\delta,N,L}$. We denote this periodic extension by f. We define

$$C_{L,N}=C_{L\left(1-\frac{2}{N}\right)}.$$

So $C_{L,N}$ is the sub-cube of C_L with the outer layer of cubes of size L/N removed. Let

$$\tilde f=\left(1-\frac{\operatorname{dist}(x,C_{L,N})}{L/(4N)}\right)_+f.$$

Then $\tilde{f} \in H_0^1(\Omega_{\delta,N,L})$, and

$$\|\tilde{f}\|^2_{\mathcal{L}^2(\Omega_{\delta,N,L})} \geq \int_{C_{L,N}} f^2 = (N-2)^m, \tag{9}$$

since f restricted to any of the cubes Q_i in $\Omega_{\delta,N,L}$ is normalised. Furthermore

$$\begin{aligned} \left|D\tilde{f}\right|^2 &\leq \left(1 - \frac{\mathrm{dist}(x, C_{L,N})}{L/(4N)}\right)^2 |Df|^2 + 1_{C_L - C_{L,N}} \left(\left(\frac{4N}{L}\right)^2 f^2 + \frac{8N}{L} f\,|Df|\right) \\ &\leq |Df|^2 + \left(\frac{4N}{L}\right)^2 1_{C_L - C_{L,N}} f^2 + \frac{8N}{L} 1_{C_L - C_{L,N}} f |Df|. \end{aligned}$$

Hence

$$\begin{aligned} \int_{\Omega_{\delta,N,L}} |D\tilde{f}|^2 &\leq \int_{\Omega_{\delta,N,L}} |Df|^2 + \left(\frac{4N}{L}\right)^2 \int_{C_L - C_{L,N}} f^2 \\ &\quad + \frac{8N}{L} \left(\int_{C_L - C_{L,N}} |Df|^2\right)^{1/2} \left(\int_{C_L - C_{L,N}} f^2\right)^{1/2} \\ &= N^m \mu_{1,B(0;\delta),L/N} + \left(N^m - (N-2)^m\right) \left(\left(\frac{4N}{L}\right)^2 + \frac{8N}{L} \left(\mu_{1,B(0;\delta),L/N}\right)^{1/2}\right) \\ &\leq N^m \mu_{1,B(0;\delta),L/N} + \left(N^m - (N-2)^m\right) \left(\left(\frac{4N}{L}\right)^2 + 8N^{1/2} \mu_{1,B(0;\delta),L/N}\right), \end{aligned} \tag{10}$$

where we have used the last hypothesis in the lemma. By (9), (10), the Rayleigh-Ritz variational formula, and the hypothesis $N \geq 10$,

$$\begin{aligned} \lambda\left(\Omega_{\delta,N,L}\right) &\leq \mu_{1,B(0;\delta),L/N} \\ &\quad + \frac{N^m - (N-2)^m}{(N-2)^m} \left(\left(\frac{4N}{L}\right)^2 + \left(8N^{1/2} + 1\right) \mu_{1,B(0;\delta),L/N}\right) \\ &\leq \mu_{1,B(0;\delta),L/N} + 32m \left(\frac{5}{4}\right)^m \left(\frac{N}{L^2} + \frac{1}{N^{1/2}} \mu_{1,B(0;\delta),L/N}\right). \end{aligned} \tag{11}$$

□

To obtain an upper bound for $\|v_{\Omega_{\delta,N,L}}\|_{\mathcal{L}^\infty(\Omega_{\delta,N,L})}$, we change the Dirichlet boundary conditions on ∂C_L to Neumann boundary conditions. This increases the corresponding heat kernel, torsion function, and $\mathcal{L}^\infty$ norm respectively. By periodicity, we have that

$$\left\|v_{\Omega_{\delta,N,L}}\right\|_{\mathcal{L}^\infty(\Omega_{\delta,N,L})} \leq \|\tilde{v}_{C_{L/N} - B(0;\delta)}\|_{\mathcal{L}^\infty\left(C_{L/N} - B(0;\delta)\right)}, \tag{12}$$

where $\tilde{v}_{C_{L/N} - B(0;\delta)}$ is the torsion function with Neumann boundary conditions on $\partial C_{L/N}$, and Dirichlet boundary conditions on $\partial B(0;\delta)$. Denote the spectrum of the corresponding Laplacian by $\{\mu_j := \mu_{j,B(0;\delta),L/N}, j = 1, 2, \dots\}$, and let $\{\varphi_j := \varphi_{1,B(0;\delta),L/N}, j = 1, 2, \dots\}$ denote a corresponding orthonormal basis of eigenfunctions. We denote by $\pi_{\delta,N/L}(x,y;t), x \in C_{L/N} - B(0;\delta), y \in C_{L/N} - B(0;\delta), t > 0$ the corresponding heat kernel. Then

$$\pi_{\delta,N/L}(x,y;t) = \sum_{j=1}^{\infty} e^{-t\mu_j} \varphi_j(x)\varphi_j(y), \tag{13}$$

and

$$\begin{aligned}
&\tilde{v}_{C_{L/N}-B(0;\delta)}(x) \\
&\quad = \int_0^\infty dt \int_{C_{L/N}-B(0;\delta)} dy\ \pi_{\delta,N/L}(x,y;t)\left(\frac{\varphi_1(y)}{\|\varphi_1\|} + 1 - \frac{\varphi_1(y)}{\|\varphi_1\|}\right) \\
&\quad = \frac{1}{\mu_1}\frac{\varphi_1(x)}{\|\varphi_1\|} + \int_0^\infty dt \int_{C_{L/N}-B(0;\delta)} dy\ \pi_{\delta,N/L}(x,y;t)\left(1 - \frac{\varphi_1(y)}{\|\varphi_1\|}\right) \\
&\quad \le \frac{1}{\mu_1} + \int_0^T dt \int_{C_{L/N}-B(0;\delta)} dy\ \pi_{\delta,N/L}(x,y;t) \\
&\qquad + \int_T^\infty dt \int_{C_{L/N}-B(0;\delta)} dy\ \pi_{\delta,N/L}(x,y;t)\left(1 - \frac{\varphi_1(y)}{\|\varphi_1\|}\right) \\
&\quad \le \frac{1}{\mu_1} + T + \int_T^\infty dt \int_{C_{L/N}-B(0;\delta)} dy\ \pi_{\delta,N/L}(x,y;t)\left(1 - \frac{\varphi_1(y)}{\|\varphi_1\|}\right),
\end{aligned} \tag{14}$$

where $\|\varphi_1\| = \|\varphi_1\|_{\mathcal{L}^\infty(C_{L/N}-B(0;\delta))}$. By (13), we have that the third term in the right-hand side of (14) equals

$$\sum_{j=1}^{\infty} \mu_j^{-1} e^{-T\mu_j} \varphi_j(x) \int_{C_{L/N}-B(0;\delta)} dy\ \varphi_j(y)\left(1 - \frac{\varphi_1(y)}{\|\varphi_1\|}\right). \tag{15}$$

The term with $j = 1$ in (15) is bounded from above by

$$\begin{aligned}
\mu_1^{-1}\|\varphi_1\| &\int_{C_{L/N}-B(0;\delta)} \|\varphi_1\|\left(1 - \frac{\varphi_1}{\|\varphi_1\|}\right) \\
&= \mu_1^{-1}\|\varphi_1\| \int_{C_{L/N}-B(0;\delta)} \left(\|\varphi_1\| - \varphi_1\right) \\
&\le \mu_1^{-1}\left(\|\varphi_1\|^2\left(\frac{L}{N}\right)^m - 1\right),
\end{aligned}$$

where we used the fact that $1 = \int_{C_{L/N}-B(0;\delta)} \varphi_1^2 \le \|\varphi_1\| \int_{C_{L/N}-B(0;\delta)} \varphi_1$. It was shown on p.586, lines -3,-4, in [9] (with appropriate adjustment in notation) that

$$\|\varphi_1\|^2 \le \left(\frac{N}{L}\right)^m \left(1 - s\mu_1 - \frac{mL^2}{3esN^2}\right)^{-1}, \quad s \ge 0,$$

provided the last term in the round brackets is non-negative. The optimal choice for s gives that

$$\|\varphi_1\|^2 \le \left(\frac{N}{L}\right)^m \left(1 - \frac{(4m\mu_1)^{1/2}L}{(3e)^{1/2}N}\right)^{-1}, \quad \mu_1 < \frac{3eN^2}{4mL^2}.$$

By further restricting the range for μ_1, we have that the first term with $j = 1$ in (15) is then bounded from above by

$$\mu_1^{-1} \frac{2L\left(m\mu_1/(3eN^2)\right)^{1/2}}{1-2L\left(m\mu_1/(3eN^2)\right)^{1/2}} \le \frac{(2m)^{1/2}L}{\mu_1^{1/2}N}, \qquad \mu_1 \le \frac{3eN^2}{16mL^2}. \tag{16}$$

The terms with $j \ge 2$ in (15) give, by Cauchy–Schwarz for both the series in j, and the integral over $C_{L/N} - B(0;\delta)$, a contribution

$$\begin{aligned}
&\left|\sum_{j=2}^{\infty} \mu_j^{-1} e^{-T\mu_j} \varphi_j(x) \int_{C_{L/N}-B(0;\delta)} \varphi_j \left(1 - \frac{\varphi_1}{\|\varphi_1\|}\right)\right| \\
&\quad \le \mu_2^{-1} \sum_{j=2}^{\infty} e^{-T\mu_j} \left|\varphi_j(x)\right| \int_{C_{L/N}-B(0;\delta)} |\varphi_j| \\
&\quad \le \mu_2^{-1} \left(\frac{L}{N}\right)^{m/2} \left(\sum_{j=2}^{\infty} e^{-T\mu_j}\right)^{1/2} \left(\sum_{j=2}^{\infty} e^{-T\mu_j} \left|\varphi_j(x)\right|^2\right)^{1/2} \\
&\quad \le \mu_2^{-1} \left(\frac{L}{N}\right)^{m/2} \left(\sum_{j=2}^{\infty} e^{-T\mu_j}\right)^{1/2} \left(\pi_{\delta,N/L}(x,x;T)\right)^{1/2}.
\end{aligned} \tag{17}$$

To bound the first series in the right-hand side of (17), we note that the μ_j's are bounded from below by the Neumann eigenvalues of the cube $C_{L/N}$. So choosing $T = (L/N)^2$ we get that

$$\left(\sum_{j=2}^{\infty} e^{-L^2\mu_j/N^2}\right)^{1/2} \le \left(1 + \sum_{j=1}^{\infty} e^{-\pi^2 j^2}\right)^{m/2} \le \left(\frac{4}{3}\right)^{m/2}.$$

Similarly to the proof of Lemma 3.1 in [9], we have that

$$\begin{aligned}
\left(\pi_{\delta,N/L}\left(x,x;L^2/N^2\right)\right)^{1/2} &\le \left(\pi_{0,N/L}\left(x,x;L^2/N^2\right)\right)^{1/2} \\
&\le \left(\frac{N}{L}\right)^{m/2} \left(1 + 2\sum_{j=1}^{\infty} e^{-\pi^2 j^2}\right)^{m/2} \\
&\le \left(\frac{4}{3}\right)^{m/2} \left(\frac{N}{L}\right)^{m/2}.
\end{aligned} \tag{18}$$

Finally, $\mu_2 \ge \frac{\pi^2 N^2}{L^2}$, together with (12), (14), (16), (17), (18), and the choice $T = (L/N)^2$ gives that

$$\left\|v_{\Omega_{\delta,N,L}}\right\|_{\mathcal{L}^\infty(\Omega_{\delta,N,L})} \le \mu_1^{-1} + \frac{(2m)^{1/2}L}{\mu_1^{1/2}N} + \left(\frac{4}{3}\right)^m \frac{L^2}{N^2}, \qquad \mu_1 \le \frac{3eN^2}{16mL^2}. \tag{19}$$

Proof of Theorem 1. Let $1<\alpha<2$. By (11) and (19), we have that

$$\lambda\left(\Omega_{\delta,N,L}\right)\left\|v_{\Omega_{\delta,N,L}}\right\|_{\mathcal{L}^\infty(\Omega_{\delta,N,L})}\le\left(\mu_1+32m\left(\frac{5}{4}\right)^m\left(\frac{N}{L^2}+\frac{1}{N^{1/2}}\mu_1\right)\right)\\ \times\left(\mu_1^{-1}+\frac{(2m)^{1/2}L}{\mu_1^{1/2}N}+\left(\frac{4}{3}\right)^m\frac{L^2}{N^2}\right), \tag{20}$$

provided

$$\frac{N}{L^2}\le\mu_1\le\frac{3eN^2}{16mL^2}.$$

First consider the planar case $m=2$. Recall Lemma 3.1 in [9]: for $\delta<L/(6N)$,

$$\frac{N^2}{100L^2}\left(\log\frac{L}{2\delta N}\right)^{-1}\le\mu_{1,B(0;\delta),L/N}\le\frac{8\pi N^2}{(4-\pi)L^2}\left(\log\frac{L}{2\delta N}\right)^{-1}. \tag{21}$$

Let

$$\delta^*:=\delta^*(\alpha,N,L)=\frac{L}{2N}e^{-N^{2-\alpha}}, \tag{22}$$

where $1<\alpha<2$. Let $N_1\in\mathbb{N}$ be such that for all $N\ge N_1$, $\delta^*<L/(6N)$. We now use (21) to see that there exists $C>1$ such that

$$C^{-1}\frac{N^\alpha}{L^2}\le\mu_{1,B(0;\delta^*),L/N}\le C\frac{N^\alpha}{L^2}. \tag{23}$$

(In fact $C=\max\{100,8\pi/(4-\pi)\}$). We subsequently let $N_2\in\mathbb{N}$ be such that for all $N\ge N_2$,

$$\frac{N}{L^2}\le C^{-1}\frac{N^\alpha}{L^2}\le C\frac{N^\alpha}{L^2}\le\frac{3eN^2}{16mL^2}.$$

By (20), (23), and all $N\ge\max\{N_1,N_2\}$ we have that

$$\lambda(\Omega_{\delta^*,N,L})\|v_{\Omega_{\delta^*,N,L}}\|_{\mathcal{L}^\infty(\Omega_{\delta^*,N,L})}\le 1+\mathcal{C}\big(N^{1-\alpha}+N^{(\alpha-2)/2}\big), \tag{24}$$

where $\mathcal{C}$ depends on C and on m only. Finally, we let $N_3\in\mathbb{N}$ be such that for all $N\ge N_3$,

$$\mathcal{C}\big(N^{1-\alpha}+N^{(\alpha-2)/2}\big)<\epsilon.$$

We conclude that (5) holds with $\Omega_\epsilon=\Omega_{\delta^*,N,L}$ with δ^* given by (22), and $N\ge\max\{N_1,N_2,N_3\}$.

Next consider the case $m=3,4,\dots$. We apply Lemma 3.2 in [9] to the case $K=B(0;\delta)$, and denote the Newtonian capacity of K by $\mathrm{cap}(K)$. Then $\mathrm{cap}(B(0;\delta))=\kappa_m\delta^{m-2}$, where κ_m is the Newtonian capacity of the ball with radius 1 in $\mathbb{R}^m$. Then Lemma 3.2 gives that there exists $C\ge 1$ such that

$$C^{-1}\left(\frac{N}{L}\right)^m\delta^{m-2}\le\mu_{1,B(0;\delta),L/N)}\le C\left(\frac{N}{L}\right)^m\delta^{m-2}, \tag{25}$$

provided

$$\kappa_m\delta^{m-2}\le\frac{1}{16}(L/N)^{m-2}. \tag{26}$$

We choose

$$\delta^* := \delta^*(\alpha, N, L) = LN^{(\alpha-m)/(m-2)}. \tag{27}$$

This gives inequality (23) by (25). The requirement (26) holds for all $N \geq N_1$, where N_1 is the smallest natural number such that $N_1^{2-\alpha} \geq 16\kappa_m$. The remainder of the proof follows the lines below (23) with the appropriate adjustment of constants, and the choice of δ^* as in (27). □

We note that the choice $\alpha = \frac{4}{3}$ in either (22) or in (27) gives, by (24), the decay rate

$$\lambda(\Omega_{\delta^*,N,L}) \left\| v_{\Omega_{\delta^*,N,L}} \right\|_{\mathcal{L}^\infty\left(\Omega_{\delta^*,N,L}\right)} \leq 1 + 2\mathcal{C}N^{-1/3}. \tag{28}$$

3. Proof of Theorem 2

In view of Payne's inequality (6) it suffices to obtain an upper bound for $\|v_\Omega\|_{\mathcal{L}^\infty(\Omega)}\lambda(\Omega)$. We first observe, that by domain monotonicity of the torsion function, v_Ω is bounded by the torsion function for the (connected) set bounded by the two parallel lines tangent to Ω at distance $w(\Omega)$. Hence

$$\|v_\Omega\|_{\mathcal{L}^\infty(\Omega)} \leq \frac{w(\Omega)^2}{8}. \tag{29}$$

In order to obtain an upper bound for $\lambda(\Omega)$, we introduce the following notation. For a planar, open, convex set, with finite measure, we let z_1, z_2 be two points on the boundary of Ω which realise the width. That is there are two parallel lines tangent to $\partial\Omega$, at z_1 and z_2 respectively, and at distance $w(\Omega)$. Let the x-axis be perpendicular to the vector $z_1 z_2$, containing the point $\frac{1}{2}(z_1 + z_2)$. We consider the family of line segments parallel to the x-axis, obtained by intersection with Ω, and let l_1, l_2 be two points on the boundary of Ω which realise the maximum length L of this family. The quadrilateral with vertices, z_1, z_2, l_1, l_2 is contained in Ω. This quadrilateral in turn contains a rectangle with side-lengths h, and $\left(1 - \frac{h}{w(\Omega)}\right)L$ respectively, where $h \in [0, w(\Omega))$ is arbitrary. Hence, by domain monotonicity of the Dirichlet eigenvalues, we have that

$$\lambda(\Omega) \leq \pi^2 h^{-2} + \pi^2 \left(1 - \frac{h}{w(\Omega)}\right)^{-2} L^{-2}.$$

Minimising the right-hand side above with respect to h gives that

$$h = \frac{\left(w(\Omega)L^2\right)^{1/3}}{1 + \left(\frac{L}{w(\Omega)}\right)^{2/3}}.$$

It follows that

$$\lambda(\Omega) \leq \frac{\pi^2}{w(\Omega)^2}\left(1 + \left(\frac{w(\Omega)}{L}\right)^{2/3}\right)^3. \tag{30}$$

As $w(\Omega) \le L$ we obtain by (30) that

$$\lambda(\Omega) \le \frac{\pi^2}{w(\Omega)^2}\left(1 + 7\left(\frac{w(\Omega)}{L}\right)^{2/3}\right). \tag{31}$$

In order to complete the proof we need the following.

Lemma 5. *If Ω is an open, bounded, convex set in $\mathbb{R}^2$, and if L is the length of the longest line segment in the closure of Ω, perpendicular to z_1z_2, then*

$$\mathrm{diam}(\Omega) \le 3L. \tag{32}$$

Proof. Let $d_1, d_2 \in \partial\Omega$ such that $|d_1 - d_2| = \mathrm{diam}(\Omega)$. We denote the projections of d_1, d_2 onto the line through z_1, z_2 by e_1, e_2 respectively. Let z be the intersection of the lines through z_1, z_2 and d_1, d_2 respectively. Then, by the maximality of L, we have that $|d_1 - e_1| \le L, |d_2 - e_2| \le L$. Furthermore, by convexity, $|e_1 - z| + |e_2 - z| \le w(\Omega)$. Hence,

$$|d_1 - d_2| \le |d_1 - e_1| + |e_1 - z| + |d_2 - e_2| + |e_2 - z| \le 2L + w(\Omega) \le 3L.$$

□

By (31), we have that

$$\lambda(\Omega) \le \frac{\pi^2}{w(\Omega)^2}\left(1 + 7 \cdot 3^{2/3}\left(\frac{w(\Omega)}{\mathrm{diam}(\Omega)}\right)^{2/3}\right).$$

This implies Theorem 2 by (29). □

4. Proof of Theorem 3

We denote by $d : M \times M \mapsto \mathbb{R}^+$ the geodesic distance associated to (M, g). For $x \in M$, $R > 0$, $B(x;R) = \{y \in M : d(x,y) < R\}$. For a measurable set $A \subset M$ we denote by $|A|$ its Lebesgue measure. The Bishop–Gromov Theorem (see [10]) states that if M is a complete, non-compact, m-dimensional, Riemannian manifold with non-negative Ricci curvature, then for $p \in M$, the map $r \mapsto \frac{|B(p;r)|}{r^m}$ is monotone decreasing. In particular

$$\frac{|B(p;r_2)|}{|B(p;r_1)|} \le \left(\frac{r_2}{r_1}\right)^m,\ 0 < r_1 \le r_2. \tag{33}$$

Corollary 3.1 and Theorem 4.1 in [11], imply that if M is complete with non-negative Ricci curvature, then for any $D_2 > 2$ and $0 < D_1 < 2$ there exist constants $0 < C_1 \le C_2 < \infty$ such that for all $x \in M,\ y \in M,\ t > 0$,

$$C_1 \frac{e^{-d(x,y)^2/(2D_1 t)}}{\left(|B(x;t^{1/2})||B(y;t^{1/2})|\right)^{1/2}} \le p_M(x,y;t) \le C_2 \frac{e^{-d(x,y)^2/(2D_2 t)}}{\left(|B(x;t^{1/2})||B(y;t^{1/2})|\right)^{1/2}}. \tag{34}$$

Finally, since by (33) the measure of any geodesic ball with radius r is bounded polynomially in r, the theorems of Grigor'yan in [6] imply stochastic completeness. That is, for all $x \in M$, and all $t > 0$,

$$\int_M dy\, p_M(x, y; t) = 1.$$

Proof of Theorem 3. We choose $D_1 = 1$, $D_2 = 3$ in (34), and define the corresponding number $K = \max\{C_2, C_1^{-1}\}$. Then

$$K^{-1} e^{-d(x,y)^2/(2t)} \le \Big(|B(x; t^{1/2})||B(y; t^{1/2})|\Big)^{1/2} p_M(x, y; t) \le K e^{-d(x,y)^2/(6t)}. \tag{35}$$

Let $q \in M$ be arbitrary, and let $R > 0$ be such that $\Omega(q; R) := B(q; R) \cap \Omega \neq \emptyset$. The spectrum of the Dirichlet Laplacian acting in $L^2(\Omega(q; R))$ is discrete. Denote the bottom of this spectrum by $\lambda(\Omega(q; R))$. Then $\lambda(\Omega(q; R)) \ge \lambda(\Omega)$. By the spectral theorem, monotonicity of Dirichlet heat kernels, and the Li-Yau bound (35), we have that

$$\begin{aligned} p_{\Omega(q;R)}(x, x; t) &\le e^{-t\lambda(\Omega(q;R))/2} p_{\Omega(q;R)}(x, x; t/2) \\ &\le e^{-t\lambda(\Omega(q;R))/2} p_M(x, x; t/2) \\ &\le K e^{-t\lambda(\Omega(q;R))/2} |B(x; (t/2)^{1/2})|^{-1}. \end{aligned} \tag{36}$$

By the semigroup property and the Cauchy–Schwarz inequality, for any open set $\Omega \subset M$, we have that

$$\begin{aligned} p_\Omega(x, y; t) &= \int_\Omega dz\, p_\Omega(x, z; t/2)\, p_\Omega(z, y; t/2) \\ &\le \left(\int_\Omega dz\, p_\Omega^2(x, z; t/2)\right)^{1/2} \left(\int_\Omega dz\, p_\Omega^2(z, y; t/2)\right)^{1/2} \\ &= \big(p_\Omega(x, x; t)\, p_\Omega(y, y; t)\big)^{1/2}. \end{aligned} \tag{37}$$

We obtain by (36), (37) (for $\Omega = \Omega(q; R)$), and $p_{\Omega(q;R)}(x, y; t) \le p_M(x, y; t)$, that

$$\begin{aligned} p_{\Omega(q;R)}(x, y; t) &\le \big(p_{\Omega(q;R)}(x, x; t)\, p_{\Omega(q;R)}(y, y; t)\big)^{1/4} p_M(x, y; t)^{1/2} \\ &\le K^{1/2} e^{-t\lambda(\Omega(q;R))/4} \big(|B(x; (t/2)^{1/2})||B(y; (t/2)^{1/2})|\big)^{-1/4} p_M^{1/2}(x, y; t). \end{aligned} \tag{38}$$

By (38) and (35), we have that

$$\begin{aligned} p_{\Omega(q;R)}(x, y; t) \le K e^{-t\lambda(\Omega(q;R))/4} &\big(|B(x; (t/2)^{1/2})||B(y; (t/2)^{1/2})|\big)^{-1/4} \\ &\times \big(|B(x; t^{1/2})||B(y; t^{1/2})|\big)^{-1/4} e^{-d(x,y)^2/(12t)}. \end{aligned} \tag{39}$$

By the Li-Yau lower bound in (35), we can rewrite the right-hand side of (39) to yield,

$$p_{\Omega(q;R)}(x,y;t) \le K^2 e^{-t\lambda(\Omega(q;R))/4} p_M(x,y;6t) \\ \times \frac{\left(|B(x;(6t)^{1/2})||B(y;(6t)^{1/2})|\right)^{1/2}}{(|B(x;(t/2)^{1/2})||B(y;(t/2)^{1/2})||B(x;t^{1/2})||B(y;t^{1/2})|)^{1/4}}. \tag{40}$$

By Bishop–Gromov (33), we have that the volume quotients in the right-hand side of (40) are bounded by $2^{3m/4} \cdot 3^{m/2}$ uniformly in x and y. Hence

$$p_{\Omega(q;R)}(x,y;t) \le 2^{3m/4} \cdot 3^{m/2} K^2 e^{-t\lambda(\Omega(q;R))/4} p_M(x,y;6t).$$

Since manifolds with non-negative Ricci curvature are stochastically complete, we have that

$$\int_{\Omega(q;R)} dy\, p_{\Omega(q;R)}(x,y;t) \le 2^{3m/4} \cdot 3^{m/2} K^2 e^{-t\lambda(\Omega(q;R))/4} \int_M dy\, p_M(x,y;6t) \\ = 2^{3m/4} \cdot 3^{m/2} K^2 e^{-t\lambda(\Omega(q;R))/4}.$$

Integrating the inequality above with respect to t over $[0,\infty)$ yields,

$$v_{\Omega(q;R)}(x) \le 2^{(3m+8)/4} \cdot 3^{m/2} K^2 \lambda(\Omega(q;R))^{-1} \le 2^{(3m+8)/4} \cdot 3^{m/2} K^2 \lambda(\Omega)^{-1}.$$

Finally letting $R \to \infty$ in the left-hand side above yields the right-hand side of (7).

The proof of the left-hand side of (7) is similar to the one in Theorem 5.3 in [1] for Euclidean space. We have that

$$v_{\Omega(q;R)}(x) = \int_0^\infty dt \int_{\Omega(q;R)} dy\, p_{\Omega(q;R)}(x,y;t). \tag{41}$$

We first observe that $|\Omega(q;R)| < \infty$, and so the spectrum of the Dirichlet Laplacian acting in $L^2(\Omega(q;R))$ is discrete and is denoted by $\{\lambda_j(\Omega(q;R)), j \in \mathbb{N}\}$, with a corresponding orthonormal basis of eigenfunctions $\{\varphi_{j,\Omega(q;R)}, j \in \mathbb{N}\}$. These eigenfunctions are in $\mathcal{L}^\infty(\Omega(q;R))$. Then, by (41) and the eigenfunction expansion of the Dirichlet heat kernel for $\Omega(q;R)$, we have that

$$\begin{aligned} v_{\Omega(q;R)}(x) &\ge \int_0^\infty dt \int_{\Omega(q;R)} dy\, p_{\Omega(q;R)}(x,y;t) \frac{\varphi_{1,\Omega(q;R)}(y)}{\|\varphi_{1,\Omega(q;R)}\|_{\mathcal{L}^\infty(\Omega(q;R))}} \\ &= \int_0^\infty dt\, e^{-t\lambda_1(\Omega(q;R))} \frac{\varphi_{1,\Omega(q;R)}(x)}{\|\varphi_{1,\Omega(q;R)}\|_{\mathcal{L}^\infty(\Omega(q;R))}} \\ &= \lambda_1(\Omega(q;R))^{-1} \frac{\varphi_{1,\Omega(q;R)}(x)}{\|\varphi_{1,\Omega(q;R)}\|_{\mathcal{L}^\infty(\Omega(q;R))}}. \end{aligned} \tag{42}$$

First taking the supremum over all $x \in \Omega(q;R)$ in the left-hand side of (42), and subsequently taking the supremum over all such x in the right-hand side of (42) gives

$$\left\| v_{\Omega(q;R)} \right\|_{\mathcal{L}^\infty(\Omega(q;R))} \geq \lambda(\Omega(q;R))^{-1}. \tag{43}$$

Observe that the torsion function is monotone increasing in R. Taking the limit $R \to \infty$ in the left-hand side of (43), and subsequently in the right-hand side of (43) completes the proof. □

Acknowledgements

MvdB acknowledges support by The Leverhulme Trust through International Network Grant *Laplacians, Random Walks, Bose Gas, Quantum Spin Systems.*

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Received: January 9, 2017.
Revised: March 30, 2017.